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Abstract

This short communication complements the DEA model proposed by Lovell and Pastor (Eur. J. Oper. Res. 118 (1999), 46-51), by incorporating both positive and negative criteria in the model. As such, we propose a DEA model, known as pure DEA, using a directional distance function approach.

Keywords: data envelopment analysis; invariance; directional distance function.

1. Introduction

Decisions are an integral part of human life. Regardless of the variety of problems that need to be solved, it usually happens that one must evaluate several alternatives and then choose among them. Data Envelopment Analysis (DEA) facilitates such an analysis, and, furthermore, it allows the ordering of the alternatives (i.e., decision-making units, DMUs).

By definition, DEA models have inputs and outputs; pure DEA refers to a class of models wherein either inputs only or outputs only are considered (Lovell and Pastor, 1997). Lovell and

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Pastor (1999) showed that a constant returns-to-scale model without inputs (or without outputs) is meaningless; thus, the intensity variables need to be constrained, i.e., $\sum_{k=1}^{K} z_k = 1$. Hence, they considered radial DEA models with a single constant input (output) and radial DEA models without inputs (or without outputs) so as to accommodate situations that arise in some multi-criteria decision-making problems. These models admit only positive (or negative) criteria. This means that either all of the criteria under evaluation are positive or negative.

However, in some situations one may encounter both positive and/or negative evaluations [variables], which warrant the translation invariance property to be satisfied. Furthermore, these situations turn out to be quite frequent in real life. For instance, a personnel selection problem needs the evaluation of some of the following positive (+) and negative (-) criteria: years of experience (+), level of qualification (+), and salary (-). Similarly, a vacation destination problem might be assessed based on the criteria: entertainment options (+), facilities (+), travel cost (-), and accommodation cost (-). An automobile selection problem is yet another situation in which criteria such as price (-), mileage (+), and quality (+) could be considered for evaluation. The same kind of situations may also arise in the development of indexes of social and economic development, such as the doing business index and the regional competitiveness index, among others.

In this context, the present paper complements the work done by Lovell and Pastor (1999), by incorporating both positive and negative criteria in the model.

2. Modeling

The input-oriented model in line with Lovell and Pastor (1999) can be written as follows:

$$
\begin{aligned}
&\text{Min } \lambda \\
&\text{subject to} \\
&\sum_{k=1}^{K} z_k x_{kn} \leqq \lambda x_{k_o n},\ n = 1,2,\dots,N, \\
&z_k \geqq 0,\ k = 1,2,\dots,K, \sum_{k=1}^{K} z_k = 1.
\end{aligned}
\tag{1}
$$

However, as previously mentioned, there are situations that require both positive and/or negative evaluations, which warrant the translation invariance property to be satisfied.

System (1) is unit invariant, but not translation invariant. Färe and Grosskopf (2013) showed that the following model is translation invariant:

$$
\begin{aligned}
&\text{Max } \beta \\
&\text{subject to} \\
&\sum_{k=1}^{K} z_k x_{kn} \leqq x_{k_o n} - \beta g_n,\ n = 1,2,\dots,N, \\
&\sum_{k=1}^{K} z_k y_{km} \geqq y_{k_o m},\ m = 1,2,\dots,M, \\
&z_k \geqq 0,\ k = 1,2,\dots,K, \sum_{k=1}^{K} z_k = 1,
\end{aligned}
\tag{2}
$$

where $g = (g_{1,} g_{2,} \dots, g_N) \geq 0,\ g \neq 0$ is the directional input vector which specifies the direction in which data are projected to the frontier of technology or, more precisely, as Aparicio et al. (2016) showed that the g vector must be translation invariant. When we remove the output constraints from System (2) we have a pure input-oriented model:

Max β

subject to

$$\sum_{k=1}^{K} z_k x_{kn} \leqq x_{k_o n} - \beta g_n,\ n = 1,2,...,N, \tag{3}$$

$$z_k \geqq 0,\ k = 1,2,...,K, \sum_{k=1}^{K} z_k = 1,$$

System (3) is both unit and translation invariant (See Appendix); given that g_n is exogenous to the data, then it is also invariant with respect to an affine data transformation. In order to incorporate both positive and negative evaluations in System (3), we shall split the N input constraints into two sets, say $N = N_1 + N_2$.

Let $[x^+]_{\mathrm{KxN_1}}$ be an evaluation matrix consisting of M alternatives (DMUs) and N_1 positive criteria (values) and let $[x^-]_{\mathrm{KxN_2}}$ be an evaluation matrix consisting of M alternatives (DMUs) and N_2 negative criteria. We have two options: to either transform the negative criteria (values) into positive criteria (values) or vice versa. Let us transform the positive criteria evaluation matrix $[x^+]_{\mathrm{KxN_1}}$ into negative criteria by allowing an affine transformation. Then, the n_1^{th} criteria (values) of the matrix can be transformed as $0 \leqq x_{kn_1}^{-'} = a_{n_1} x_{kn_1}^{+'} + b_{n_1}$, where a_{n_1} is negative. Taking account of both the positive and negative evaluations, System (4) can be obtained from System (3) based on the above discussion:

Max $\beta - \gamma$

subject to

$$\sum_{k=1}^{K} z_k x_{kn_1}^{-'} \geqq x_{k_o n_1}^{-'} - \gamma g_{n_1}^{'},\ n_1 = 1,2,...,N_1, \tag{4}$$

$$\sum_{k=1}^{K} z_k x_{kn_2}^{-} \leqq x_{k_o n_2}^{-} - \beta g_{n_2},\ n_2 = 1,2,...,N_2,$$

$$z_k \geqq 0,\ k = 1,2,...,K, \sum_{k=1}^{K} z_k = 1,$$

where $0 \le g_{n_1}^{'} = a_{n_1}\, g_{n_1}$ and $g = (g_1^{'}, g_2^{'}, ..., g_{N_1}^{'}, g_1, g_2, ..., g_{N2}) \ge 0,\ g \ne 0$ is the directional input vector which specifies the direction in which data are projected to the frontier of technology, and the efficiency indicator for System (4) is $\beta + \gamma$ and it is to be noted that both β and γ are unit free.

To relate our model in System (4) to the model of Lovell and Pastor in System (1): in System (4) by deleting the first inequality constraint and γ in the objective function; then, by taking g_{n_2}, $n_2 = 1,2,...,N_2$, equal to $x_{k_o n_2}^{-}$ to obtain $x_{k_o n_2}^{-}(1-\beta)$ on the right-hand-side; finally, by setting $(1-\beta) = \lambda,$ and adjusting the objective function so as to obtain $1 + \text{Min}\ \lambda,$ the relation to Lovell and Pastor's model is established.

3. Numerical Example

The following example demonstrates the applicability of the above sentences of System (4) numerically:

Let us assume that a potential costumer is looking forward to purchasing an automobile. His choice must be made among the given six alternatives (DMUs), represented below by A, B, C, D, E, and F. He needs to compare their performance characteristics in order to find the best options. The criteria considered could be: the price of the automobile (-), the mileage (+), and the quality (+). The data is as defined in the below Table 1, where the numbers provided have been chosen for reasons of simplicity of understanding. As such, we show how we can compare these automobiles and choose the best among the given alternatives.

Table 1: One negative criterion (Price) and two positive criteria (Mileage and Quality)

Alternatives (DMU) k	Price x_{k1}^{-}	Original Mileage x_{k2}^{+}	Transformed Mileage $x_{k2}^{-'} = -1.x_{k2}^{+} + 7$	Original Quality x_{k3}^{+}	Transformed Quality $x_{k3}^{-'} = -1.x_{k3}^{+} + 10$
A	10	0	7	8	2
B	5	6	1	6	4
C	8	5	2	7	3
D	3	6	1	9	1
E	**2**	**6**	**1**	8	2
F	4	4	3	9	1

Alternatives (DMU) k	β	γ	Objective Function $\beta - \gamma$	Efficiency Indicator $\beta + \gamma$
A	8	6	2	14 (6)
B	3	2	1	5 (4)
C	6	1	5	7 (5)
D	1	0	1	1 (2)
E	**0**	**0**	**0**	**0 (1)**
F	2	2	0	4 (3)

It is to be noted that in the case of multiple optimal solutions, one should select the minimum (maximum) of all the sets $\beta + \gamma$ among all the optimal solutions, under the optimistic (pessimistic) approach towards the DMU of interest. One may also think of using the weighted average of the values obtained in the optimistic and pessimistic approaches. For instance, let us assume that DMU-A has alternative optimal solutions (8, 6) and (16, 14). In such a case, the optimistic, pessimistic, and weighted average (assuming equal weights) approaches yield an efficiency indicator of 14, 30, and 22, respectively. In consequence, depending on the approach, the order of the DMUs varies, which is inevitable in multi-criteria decision-making.

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Appendix

To verify that (2) is both unit and translation invariant let us look at the nth input

$$\sum_{k=1}^{K} z_k x_{kn} \leqq x_{k'n} - \beta g_n , \tag{1'}$$

Note that g_n is exogenous to the data. Change the unit by a and translate the data by b, then

$$\sum_{k=1}^{K} z_k (ax_{kn} + b) \leqq (ax_{k'n} + b) - a\beta g_n , \tag{2'}$$

remembering that g_n has the same unit of measurement as $x_{k'n}$. Now,

$$\sum_{k=1}^{K} z_k a x_{kn} + \sum_{k=1}^{K} z_k b \leqq ax_{k'n} + b - \beta a g_n , \tag{3'}$$

and by variable returns-to-scale, $\sum_{k=1}^{K} z_k = 1$,

$$a\sum_{k=1}^{K} z_k x_{kn} \leqq a(x_{k'n} - \beta g_n), \tag{4'}$$

and since a cancels we have our original expression (1').