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Abstract – Probabilistic or stochastic programming is a framework for modeling optimization problems that involve uncertainty. Stochastic programming models arise as reformulations or extensions of reliability optimization problems with random parameters. Moreover, the resource elements vary and it is reasonable to consider them as stochastic variables. In this paper, we describe the chance-constrained reliability stochastic optimization (CCRSO) problem for which the objective is to maximize the system reliability for the given joint chance constraints where only the resource variables are random in nature and which follow different general form of distributions. Few numerical examples are also presented to illustrate the applicability of the methodology.

Keywords – Chance-constrained programming, reliability optimization, joint constraints, general form of distributions.

1. INTRODUCTION

Stochastic programming (SP) models were first formulated by Dantzig (1955) who suggested a two-stage programming technique that involves the conversion of SP models into their equivalent deterministic programming models. However, this technique suffers from the limitation that it does not allow any constraint to be violated even at a specific probability level. This gave rise to the concept of chance-constrained programming (CCP), where constraints containing random variables are guaranteed to be satisfied with a certain probability. Charnes and Cooper (1959, 1963) developed the concept of CCP. For the interested reader, notable contributions to the field can be found in Kataoka (1963), Van De Panne and Popp (1963), Charnes, Cooper, and Thompson (1964, 1965), Charnes, Kirby, and Raike (1967), Williams (1965, 1966), Naslund (1966), Wets (1966), Symonds (1967), Ziemba (1970), Lee and Olson (1985), Olson and Swenseth (1987), Seppala (1988), Shapiro (1990), Weintraub and Vera (1991), Flam and Schult (1993), Schoen (1994), Zhao and Ziemba (2001), Beraldi and Bruni (2010), Cheng and Lisser (2012, 2013), and Kucukyavuz (2012). Joint probabilistic constraints for independent random variables were used initially by Miller and Wagner (1965) and Jagannathan (1974). Charles and Dutta (2005) also derived the deterministic equivalent of the objective function and constraint coefficients with normal random variables. The properties of stochastic programming problems and methods for obtaining an optimal solution were described in Sengupta and Fox (1969), Tintner and Sengupta (1972), Vajda (1972), Rao (1989), Kall and Wallace (1994), Prékopa (1995) and Birge and Louveaux (2011). In this regards, a bibliography was provided by Stancu-Minasian and Wets (1976).

Reliability is defined as the probability that a device or system is able to perform its intended functions satisfactorily under specified conditions for a specified period of time. However, traditional reliability assumes that a system and its components can be in either a completely working or a completely failed state only (Birnbaum, Esary, & Saunders, 1961), i.e., no intermediate states are allowed. A reliability-based methodology for the robust optimal design of uncertain linear structural systems subjected to stochastic dynamic loads was also presented by Papadimitriou, Katafygiotis, and Siu (1997) and Papadimitriou and Ntotsios (2004).

Solution methods in the literature for reliability optimization of complex systems are mainly heuristic methods. In recent years, metaheuristic algorithms such as genetic algorithm (Gen & Cheng, 1997), simulated annealing (Ravi, Murty, & Reddy, 1997), and Tabu search (Glover & Laguna, 1993) have also been applied to reliability optimization of complex systems. A comprehensive review of heuristic and metaheuristic algorithms for reliability optimization can be found in a relatively recent survey paper (Kuo & Prasad, 2000). Exact methods for reliability problems include the branch-and-bound methods (Nakagawa, Nakashima, & Hattori, 1978; Tillman, Hwang, & Kuo, 1980), dynamic programming methods (Ng & Sancho, 2001; Sniedovich & Vazirinejad, 1990), and implicit enumeration methods (Misra & Sharma, 1991; Prasad & Kuo, 2000). For a more comprehensive view of the research works undergone in the field of reliability optimization for the past four decades, we refer the interested readers to the following authors, whom we found to provide useful insights: Aggarwal, Gupta, and Misra (1975), Aggarwal (1976), Tillman, Hwang, and Kuo (1977a,1977b), Kuo, Hwang, and Tillman (1978), Tillman et al. (1980), Hwang, Tillman, and Lee (1981), Painton and Campbell (1995), Sung and Cho (1999), Kuo and Prasad (2000), Ravi, Reddy, and Zimmermann (2000), Kuo, Prasad, Tillman, and Hwuang (2001), Sun, Mckinnon, and Li (2001), Sun and Li (2002), Zhao and Liu (2003), Coit, Jin, and Wattanapongsakorn (2004), Cui, Kuo, Loh, and Xie (2004), Liang and Smith (2004), Ramirez-Marquez, Coit, and Konak (2004), Zafiropoulos and Dialynas (2004), Aggarwal and Gupta (2005), Marseguerra and Podofillini (2005), Gen and Yun (2006), Ha and Kuo (2006a, 2006b), Kuo and Wan (2007), Onishi and Kimura (2007), Ramirez-Marquez and Coit (2007), Yadavalli, Malada, and Charles (2007), Zhao, Cui, and Kuo (2007), Coelho (2009), Chan and Lin (2011), Minguez, Conejo, and Garcia-Bertrand (2011), Nikolaidis and Mourelatos (2011), and Sakalli (2014).

The redundancy allocation problem (RAP) is a difficult combinatorial optimization problem (Chern, 1992). It was extensively studied in the past, and when considering binary components, it was solved as a single objective optimization problem (generally maximization of system reliability), subject to several constraints, such as cost, weight, and volume, among others. It was solved using mathematical models, such as dynamic programming (Bellman & Dreyfus, 1958; Misra, 1971; Fyffe, Hines, & Lee 1968), integer programming (Bulfin & Liu, 1985; Misra & Sharma, 1991), mixed integer and non-linear programming (Tillman et al., 1977a, 1977b), and metaheuristics, such as genetic algorithms (Coit & Smith, 1996; Ida, Gen, & Yokota, 1994; Painton & Campbell, 1995), Tabu search (Kulturel-Konak, Smith, &Coit, 2003), and ant colony optimization (Liang & Smith, 2004).

This paper is organized as follows: first, the literature review is presented in section 1. In section 2, the mathematical model of a stochastic integer programming of an n-stage series system with m-joint chance constraints problem is defined and its deterministic equivalent form is derived. Moreover, some general form of distributions and their various deductions are discussed in section 3. Few numerical examples are then presented in section 4 and section 5 concludes the paper. For the interested reader, a more detailed analysis and further information in this regards can be found in the work undergone by Ansari (2011).

2. STOCHASTIC INTEGER PROGRAMMING: *N*-STAGE SERIES SYSTEM WITH *M*-JOINT CHANCE CONSTRAINTS

The chance-constrained programming problem for an *n*-stage series system with *m*-joint chance constraints can be formulated as:

$$Max\, R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right] \tag{1}$$

subject to

$$P\left[g_1(x) \le b_1, g_2(x) \le b_2, \ldots, g_m(x) \le b_m\right] \ge p,$$

$$x_j \ge 1, \quad j = 1,2,\ldots n,$$

where

$R_s(X)$ - Reliability of the system
r_j, q_j - Reliability, unreliability of components j ; $r_j + q_j = 1$
x_j - Number of components used at stage j
$g_i(x)$ - Chance constraint i
b_i - Amount of resource i available (random)
$0 < p < 1$, usually close to 1.

The joint chance-constraints of System (1) may also be written as $\prod_{i=1}^{m} P(b_i \geq y_i) \geq p,$

where $y_i = g_i(x)$. Hence System (1) has the following form:

$$Max\, R_s(X) = \prod_{j=1}^{n} \left[1 - (1 - r_j)^{x_j}\right] \tag{2}$$

subject to

$$\prod_{i=1}^{m} P(b_i \geq y_i) \geq p,$$

$$x_j \geq 1, \quad j = 1,2,\ldots n.$$

3. VARIOUS SPECIAL CASES FOR JOINT CHANCE CONSTRAINTS

Case 1: In System (1), let b_i follows a general form of distributions $F(b_i) = 1 - [A_i h(b_i) + B_i]^{C_i}$

It is given that the i^{th} random variable b_i has three known parameters $A_i(\neq 0)$, $B_i(\geq 0)$ and $C_i(\neq 0)$ such that $F(\alpha_i) = 0, F(\beta_i) = 1$ and $h(b_i)$ is a monotonic, continuous, and differentiable function of b_i in the interval $[\alpha_i, \beta_i]$. The probability density function (pdf) of the random variable b_i is given by

$$f(b_i) = -A_i C_i [A_i h(b_i) + B_i]^{C_i - 1} h'(b_i) \tag{3}$$

Now, for the above *pdf* the joint probabilistic constraints in System (1) can be written as:

$$\prod_{i=1}^{m} \left(\int_{y_i}^{\beta_i} - A_i C_i [A_i h(b_i) + B_i]^{C_i - 1} h'(b_i)\, db_i \right) \geq p \tag{4}$$

After integration, we have:

$$\prod_{i=1}^{m}\left[A_i h(y_i)+B_i\right]^{C_i} \geq p, \quad \text{as } \left[A_i h(\beta_i)+B_i\right]^{C_i} = 0.$$

Hence, for the given random variable, the joint chance constraints System (2) are converted into joint deterministic constraints as follows:

$$\prod_{i=1}^{m}\left[A_i h(y_i)+B_i\right]^{C_i} \geq p \tag{5}$$

The deterministic constraints may get the information from the following distributions when the latter follow the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$:

Power Function distribution $\left[-\lambda_i^{-a_i}, 1, 1, b_i^{a_i}\right]$, Pareto distribution $\left[\lambda_i^{-\delta_i}, 0, -a_i\delta_i^{-1}, b_i^{a_i}\right]$, Beta distribution of first kind $\left[1, 0, a_i, (\lambda_i - b_i)(\lambda_i - \delta_i)^{-1}\right]$, Weibull distribution $\left[1, 0, \theta_i\delta_i^{-1}, e^{-\delta_i b_i^{a_i}}\right]$, Inverse Weibull distribution $\left[-1, 1, 1, e^{-\theta_i b_i^{-a_i}}\right]$, Burr Type II distribution $\left[-1, 1, 1, \left(1+e^{-b_i}\right)^{-k_i}\right]$, Burr Type III distribution $\left[-1, 1, 1, \left(1+b_i^{-\lambda_i}\right)^{-k_i}\right]$, Burr Type IV distribution $\left[-1, 1, 1, \left\{1+\left\{b_i^{-1}(\lambda_i - b_i)\right\}^{1/\lambda_i}\right\}^{-k_i}\right]$, Burr Type V distribution $\left[-1, 1, 1, \left(1+\lambda_i e^{-\tan b_i}\right)^{-k_i}\right]$, Burr Type VI distribution $\left[-1, 1, 1, \left(1+\lambda_i e^{-k_i \sinh b_i}\right)^{-k_i}\right]$, Burr Type VII distribution $\left[-2^{-k_i}, 1, 1, (1+\tanh b_i)^{k_i}\right]$, Burr Type VIII distribution $\left[-2\pi^{-1}, 1, 1, \left(\tan^{-1} e^{-b_i}\right)^{k_i}\right]$, Burr Type IX distribution $\left[0.5\lambda_i, (1-0.5\lambda_i), -1, \left(1+e^{b_i}\right)^{-k_i}\right]$, Burr Type X distribution $\left[-1, 1, 1, \left\{1-\exp\left(-b_i^2\right)\right\}^{k_i}\right]$, Burr Type XI distribution $\left[-1, 1, 1, \left\{b_i - (2\pi)^{-1}\sin 2\pi b_i\right\}^{k_i}\right]$, Burr Type XII distribution $\left[\theta_i, 1, -m_i, b_i^{a_i}\right]$, and Cauchy distribution $\left[-\pi^{-1}, 0.5, 1, \tan^{-1} b_i\right]$.

Hence, in this case, the deterministic form of the chance-constrained programming problem for *n*-stage series with *m*-joint chance constraints is given by:

$$\begin{aligned}
&Max\, R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right] \\
&subject\ to \\
&\prod_{i=1}^{m}\left[A_i h(y_i)+B_i\right]^{C_i} \geq p, \\
&x_j \geq 1, \quad j = 1, 2, \ldots n.
\end{aligned} \tag{6}$$

After inserting the particular value of the parameters $[A_i, B_i, C_i, h(b_i)]$ in the above deterministic constraints, we get different deterministic constraints for different distributions.

Case 2: In System (1), let b_i follows general form of distributions $F(b_i) = A_i[h(b_i)]^{-C_i} + B_i$

The i^{th} random variable b_i is acknowledged to have three known parameters A_i, B_i and C_i. These parameters are defined as $A_i(\neq 0)$, $B_i(\geq 0)$ and $C_i(\neq 0)$ and the following conditions apply: $F(\alpha_i) = 0, F(\beta_i) = 1$ and $h(b_i)$ is a monotonic, continuous and differentiable function of b_i in the interval $[\alpha_i, \beta_i]$. In this context, the pdf of the random variable b_i is given by:

$$f(b_i) = -A_i C_i [h(b_i)]^{-C_i - 1} h'(b_i). \tag{7}$$

Now, the joint chance constraints in System (2) for the above *pdf* can be written as:

$$\prod_{i=1}^{m} \left(\int_{y_i}^{\beta_i} - A_i C_i [h(b_i)]^{-C_i - 1} h'(b_i) db_i \right) \geq p \tag{8}$$

The integration leads to:

$$\prod_{i=1}^{m} \left\{ A_i [h(y_i)]^{-C_i} + B_i \right\} \leq 1 - p, \quad \text{as } A_i [h(\beta_i)]^{-C_i} + B_i = 0.$$

As such, for the given random variable, the following joint deterministic constraints can be obtained from the conversion of the joint chance constraints in System (2):

$$\prod_{i=1}^{m} \left\{ A_i [h(y_i)]^{-C_i} + B_i \right\} \leq 1 - p. \tag{9}$$

The deterministic constraints may obtain the information from the distributions listed below when the latter follow the parameters $[A_i, B_i, C_i, h(b_i)]$:

Power Function distribution $\left[\lambda_i^{-a_i}, 0, -a_i \delta_i^{-1}, b_i^{\delta_i} \right]$, Pareto distribution $\left[-\lambda_i^{a_i}, 1, a_i \delta_i^{-1}, b_i^{\delta_i} \right]$, Beta distribution of first kind $\left[-1, 1, -a_i \delta_i^{-1}, (1 - b_i)^{\delta_i} \right]$, Weibull distribution $\left[-1, 1, \theta_i \delta_i^{-1}, e^{\delta_i b_i^{a_i}} \right]$, Inverse Weibull distribution $\left[1, 0, \theta_i \delta_i^{-1}, e^{\delta_i b_i^{-a_i}} \right]$, Burr Type II distribution $\left[1, 0, k_i \delta_i^{-1}, (1 + e^{-b_i})^{\delta_i} \right]$, Burr Type III distribution $\left[1, 0, k_i \delta_i^{-1}, (1 + b_i^{-\lambda_i})^{\delta_i} \right]$, Burr Type IV distribution $\left[1, 0, k_i \delta_i^{-1}, \left\{ 1 + \left\{ b_i^{-1} (\lambda_i - b_i) \right\}^{1/\lambda_i} \right\}^{\delta_i} \right]$, Burr Type V distribution $\left[1, 0, k_i \delta_i^{-1}, (1 + \lambda_i e^{-\tan b_i})^{\delta_i} \right]$, Burr Type VI distribution $\left[1, 0, k_i \delta_i^{-1}, (1 + \lambda_i e^{-k_i \sin b_i})^{\delta_i} \right]$, Burr Type VII distribution $\left[2^{-k_i}, 0, -k_i \delta_i^{-1}, (1 + \tanh b_i)^{\delta_i} \right]$, Burr Type VIII distribution

$\left[\left(2\pi^{-1}\right)^{k_i}, 0, -k_i\delta_i^{-1}, \left(\tan^{-1} e^{b_i}\right)^{\delta_i}\right]$, Burr Type IX distribution $\left[-2,1,\delta_i^{-1}, \left\{\lambda_i\left(1+e^{b_i}\right)^{k_i} - \lambda_i + 2\right\}^{\delta_i}\right]$, Burr Type X distribution $\left[1, 0, -k_i\delta_i^{-1}, \left(1-e^{-b_i^2}\right)^{\delta_i}\right]$, Burr Type XI distribution $\left[1, 0, -k_i\delta_i^{-1}, \left\{b_i - (2\pi)^{-1}\sin 2\pi b_i\right\}^{\delta_i}\right]$, Burr Type XII distribution $\left[-1, 1, m_i\delta_i^{-1}, \left(1+\theta_i b_i^{a_i}\right)^{\delta_i}\right]$, and Cauchy distribution $\left[\pi^{-1}, 0.5, -\delta_i^{-1}, \left(\tan^{-1} b_i\right)^{\delta_i}\right]$.

Hence, in this case, the deterministic form of the chance-constrained programming problem for *n*-stage series with *m*-chance constraints is given by:

$$
\begin{aligned}
& Max\, R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right] \\
& subject\ to \\
& \prod_{i=1}^{m} A_i\left[h(y_i)\right]^{-C_i} + B_i \le 1-p, \\
& x_j \ge 1, \quad j = 1,2,\ldots n.
\end{aligned}
\tag{10}
$$

By introducing the particular value of the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$ in the above deterministic constraints, we then obtain different deterministic constraints for different distributions.

Case 3: In System (1) let b_i follows general form of distributions $F(b_i) = 1 - B_i e^{-A_i h(b_i)}$

In this case, it is assumed that that the i^{th} random variable b_i has two known parameters A_i, B_i. A_i and B_i are such that $A_i(\ne 0)$ and $B_i(\ne 0)$ and $F(\alpha_i) = 0, F(\beta_i) = 1$. Moreover, in the interval $\left[\alpha_i, \beta_i\right], h(b_i)$ is a monotonic, continuous, and differentiable function of b_i. The *pdf* of the random variable b_i is then defined by:

$$
f(b_i) = A_i B_i e^{-A_i h(b_i)} h'(b_i) \tag{11}
$$

Now, the below represent the joint chance constraints (2.4) for the above *pdf*:

$$
\prod_{i=1}^{m}\left(\int_{y_i}^{\beta_i} A_i B_i e^{-A_i h(b_i)} h'(b_i)\, db_i\right) \ge p \tag{12}
$$

The following is obtained after integration:

$$
\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \ge p, \quad \text{as } \left[A_i h(\beta_i) + B_i\right]^{C_i} = 0
$$

The below joint deterministic constraints are then derived from the conversion of the joint chance constraints in System (2) for the given random variable:

$$\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \geq p. \tag{13}$$

When the subsequent specified distributions follow the parameters $[A_i, B_i, h(b_i)]$, the deterministic constraints may contract the information from these distributions:

Exponential distribution $[\theta_i, 1, b_i]$, Rayleigh distribution $[\theta_i, 1, b_i^2]$, Weibull distribution $[\theta_i, 1, b_i^{a_i}]$, Pareto distribution $[a_i, \lambda_i^{a_i}, \ln(b_i)]$, Lomax distribution $[a_i, 1, \ln(1+b_i\lambda_i^{-1})]$, Beta distribution of first kind $[-a_i, 1, \ln(\lambda_i - b_i)(\lambda_i - \delta_i)^{-1}]$, Beta distribution of second kind $[1, 1, \ln(1+b_i)]$, Extreme Value I distribution $[1, 1, e^{b_i}]$, Log logistic distribution $[1, 1, \ln(1-b_i)]$, Burr Type IX distribution $[1, 1, \ln\{0.5\lambda_i(1+e^{b_i})^{k_i} - 0.5\lambda_i + 1\}]$, and Burr Type XII distribution $[\lambda_i, 1, \ln(1+\theta_i b_i^{a_i})]$.

In consequence, the chance-constrained programming problem for *n*-stage series with *m*-chance constraints can be found in its deterministic form, defined as follows:

$$\begin{aligned} &Max\, R_s(X) = \prod_{j=1}^{n} \left[1-(1-r_j)^{x_j}\right] \\ &subject\ to \\ &\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \geq p, \\ &x_j \geq 1, \quad j = 1,2,\ldots n. \end{aligned} \tag{14}$$

Last, but now least, different deterministic constraints for different distributions can be obtained after inserting the particular value of the parameters $[A_i, B_i, h(b_i)]$ in the above deterministic constraints.

4. NUMERICAL EXAMPLES

Example 1: (for case 1)

We have the following stochastic problem:

$$\left.\begin{aligned} &Max\, R_s(X) = (1-0.20^{x_1})(1-0.10^{x_2})(1-0.15^{x_3}) \\ &subject\ to \\ &P\begin{pmatrix} 4.0x_1 + 2.0x_2 + 5.0x_3 \leq b_1, 3.0x_1 + 2.0x_2 + 6.0x_3 \leq b_2, \\ 6.0x_1 + 3.0x_2 + 2.0x_3 \leq b_3 \end{pmatrix} \geq 0.90, \\ &x_j \geq 1, \quad j = 1,2,3, \end{aligned}\right\} \tag{15}$$

where b_1 follows a Weibull distribution with parameters $\theta = 1/15, a = 1/4$; b_2 follows a Beta distribution of first kind with parameters $\lambda = 10,\ a = 15,\ \delta = 5$; and b_3 follows a Power Function distribution with parameters $\lambda = 12,\ a = 10.$

The deterministic model of the above stochastic problem is as follows:

$$\left.\begin{array}{l} Max\, R_s(X) = \left(1-0.20^{x_1}\right)\left(1-0.25^{x_2}\right)\left(1-0.15^{x_3}\right) \\ subject\ to \\ \left[e^{-y_1^{1/2}/20}\right]\left[\left(\dfrac{70-y_2}{10}\right)^{1/18}\right]\left[1-\left(\dfrac{y_3}{95}\right)^{85}\right] \geq 0.90, \\ x_j \geq 1,\ \ j = 1,2,3. \end{array}\right\} \qquad (16)$$

In order to obtain the reliability of the system, the above 3-stage series with 3-chance constraints problem is solved using the LINGO software. The results show that the reliability of the system is R_s = 0.9998, at x_1 =6, x_2 =5, and x_3 =5.

Example 2: (for case 2)

This second example builds upon the following stochastic model:

$$\left.\begin{array}{l} Max\, R_s(X) = \left(1-0.10^{x_1}\right)\left(1-0.15^{x_2}\right)\left(1-0.05^{x_3}\right) \\ subject\ \ to \\ P\begin{pmatrix} 4.0x_1 + 7.0x_2 + 8.0x_3 \leq b_1, 5.0x_1 + 4.0x_2 + 7.0x_3 \leq b_2, \\ 7.0x_1 + 3.0x_2 + 8.0x_3 \leq b_3, 6.0x_1 + 4.0x_2 + 2.0x_3 \leq b_4 \end{pmatrix} \geq 0.90, \\ x_j \geq 1,\ j = 1,2,3, \end{array}\right\} \qquad (17)$$

where, in this case, b_1 follows a Pareto distribution with parameters $\lambda = 10,\ a = 5$; furthermore, b_2 follows a Weibull distribution with parameters $\theta = 1/20,\ a = 1/3$; b_3 follows a Burr type IV distribution with parameters $k = 6,\ \lambda = 10$; and b_4 follows a Burr type IX distribution with parameters $\lambda = 1/20,\ k = 1/3$.

In this case, the deterministic model of the above stochastic problem becomes as follows:

$$
\left.
\begin{array}{l}
Max\, R_s(X) = \left(1-0.10^{x_1}\right)\left(1-0.15^{x_2}\right)\left(1-0.20^{x_3}\right) \\
subject\ to \\
\left[1-\left(\dfrac{60}{y_1}\right)^{1/5}\right]\left[1-e^{-\left(y_2^{1/10}/20\right)}\right]\left[\dfrac{1}{1+\left\{(75/y_3)-1\right\}^{1/75}}\right]\left[107-\dfrac{2\left(1+e^{y_4}\right)^{1/15}}{55}\right] \le 0.05, \\
x_j \ge 1, \quad j = 1,2,3.
\end{array}
\right\} \quad (18)
$$

We now have a 3-stage series but with 4-chance constraints problem that we solve by means of employing the LINGO software and we obtain the reliability of the system, that is, R_s = 0.9983, at x_1 =3, x_2 =4, and x_3 =3.

Example 3: (for case 3)

For our final example, we have the following stochastic problem:

$$
\left.
\begin{array}{l}
Max\, R_s(X) = \left(1-0.15^{x_1}\right)\left(1-0.05^{x_2}\right)\left(1-0.07^{x_3}\right)\left(1-0.10^{x_4}\right)\left(1-0.12^{x_5}\right) \\
subject\ to \\
P\left(\begin{array}{l} 3.0x_1+5.0x_2+4.0x_3+6.0x_4+2.0x_5 \le b_1,\, 2.0x_1+4.0x_2+4.0x_3+5.0x_4+6.0x_5 \le b_2, \\ 4.0x_1+3.0x_2+7.0x_3+2.0x_4+5.0x_5 \le b_3,\, 1.0x_1+2.0x_2+5.0x_3+3.0x_4+4.0x_5 \le b_4 \end{array}\right) \ge 0.99, \\
x_j \ge 1, \quad j = 1,2,3,4,5.
\end{array}
\right\} \quad (19)
$$

where b_1 follows a Burr type XII distribution with parameters $\lambda = 1/15, \theta = 1/10, a = 2/9$; b_2 , on the other hand, follows a Beta distribution of first kind with parameters $\lambda = 15,\ a = 10,\ \delta = 7$; b_3 follows a Lomax distribution with parameters $\lambda = 6,\ a = 1/5$; and, finally, b_4 follows a Pareto distribution with parameters $\lambda = 9,\ a = 5$.

The transformation of the above stochastic problem leads to obtaining the following deterministic problem:

$$
\left.
\begin{array}{l}
Max\, R_s(X) = \left(1-0.20^{x_1}\right)\left(1-0.25^{x_2}\right)\left(1-0.15^{x_3}\right)\left(1-0.10^{x_4}\right)\left(1-0.25^{x_5}\right) \\
subject\ to \\
\left[1\Big/\left(1+\dfrac{y_2^6}{12}\right)^{1/18}\right]\left[1\Big/\left(\dfrac{85-y_3}{10}\right)^{60}\right]\left[1\Big/\left(1+\dfrac{y_4}{40}\right)^{1/12}\right]\left[\left(\dfrac{65}{y_5}\right)^{1/15}\right] \ge 0.99, \\
x_j \ge 1, \quad j = 1,2,3,4,5.
\end{array}
\right\} \quad (20)
$$

The LINGO software is used once again to solve the above 5-stage series with 4-chance constraints problem and the reliability of the system is obtained, that is, R_s = 0.9979, at x_1 =5, x_2 =3, x_3 =7, x_4 =4 and x_5 =3.

5. CONCLUSION

In this paper, we formulate the chance-constrained reliability stochastic optimization problem for optimal solution to an n-stage series system with m-joint chance constraints in which only resource variables are random in nature. Various cases have been discussed with different general form of distributions when resource variables are random in nature and have different general form of distributions. After formulating the problem, we solved it using the LINGO software. One of the limitations of the study is that the current approach to tackle the problem assumes that only the right hand sides of the constraints are random in nature; simultaneously studying the case in which one can introduce randomness on the left hand side of the joint chance constraints and also in the objective function separately or combined, which is used to measure system performances such as mean system-life time, α -system lifetime and system reliability; many real life engineering problems actually do have multiple objectives, i.e., minimizing the cost, maximizing the performance, maximizing the reliability, and so on, subject to satisfying several requirements. Taking the lead from this, and in line with Charles and Udhayakumar (2012) and Charles, Undhayakumar, and Rhymend Uthariaraj (2010), the present work may be extended to multi-objective reliability optimization problems with constraints having finite probability being violated, as well as may be extended to solve the proposed systems using hybrid algorithms.

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APPENDIX

***Proposition 1*:** System (A_1) is a complement of System (6).

In System (1) if b_i follows a general form of distributions $F(b_i)=[A_i h(b_i)+B_i]^{C_i}$ then with the similar argument of Case 1, we shall obtain the following System (A_1)

$$\begin{aligned}&Max\, R_s(X)=\prod_{j=1}^{n}\left[1-(1-r_j)^{x_j}\right]\\ &subject\ to\\ &\prod_{i=1}^{m}\left[A_i h(y_i)+B_i\right]^{C_i}\le 1-p,\\ &x_j\ge 1,\quad j=1,2,\ldots n.\end{aligned} \qquad (A_1)$$

After inserting the particular value of the parameters $[A_i, B_i, C_i, h(b_i)]$ in the above deterministic constraints, we get different deterministic constraints for the various distributions listed below: Power Function distribution $\left[\lambda_i^{-\delta_i^{-1}}, 0, a_i\delta_i^{-1}, b_i^{\delta_i}\right]$, Pareto distribution $\left[-\lambda_i^{a_i}, 1, 1, b_i^{-a_i}\right]$, Beta distribution of first kind $\left[-1, 1, 1, (1-b_i)^{a_i}\right]$, Weibull distribution $\left[-1, 1, 1, \exp\left(-\theta_i b_i^{a_i}\right)\right]$, Inverse Weibull distribution $\left[1, 0, 1, e^{-\theta_i b_i^{-a_i}}\right]$, Burr Type II distribution $\left[1, 1, -k_i, e^{-b_i}\right]$, Burr Type III distribution $\left[1, 1, -k_i, b_i^{-\lambda_i}\right]$, Burr Type IV distribution , $\left[1, 1, -k_i, \left\{b_i^{-1}(\lambda_i-b_i)\right\}^{1/\lambda_i}\right]$, Burr Type V distribution , $\left[\lambda_i, 1, -k_i, e^{-\tan b_i}\right]$, Burr Type VI distribution $\left[\lambda_i, 1, -k_i, e^{-k_i \sinh b_i}\right]$, Burr Type VII distribution $[0.5, 0.5, k_i, \tanh b_i]$, Burr Type VIII distribution $\left[\left(2\pi^{-1}\right)^{\delta_i}, 0, k_i\delta_i^{-1}, \left(\tan^{-1} e^{-b_i}\right)^{\delta_i}\right]$, Burr Type IX distribution $\left[-2, 1, 1, \left\{\lambda_i\left(1+e^{b_i}\right)^{k_i}-\lambda_i+2\right\}^{-1}\right]$, Burr Type X distribution $\left[-1, 1, k_i, e^{-b_i^2}\right]$, Burr Type XI distribution $\left[1, 0, k_i\delta_i^{-1}, \left(b_i-(2\pi)^{-1}\sin 2\pi b_i\right)^{\delta_i}\right]$, Burr Type XII distribution $\left[-1, 1, 1, \left(1+\theta_i b_i^{a_i}\right)^{-m_i}\right]$, and Cauchy distribution $\left[\pi^{-1}, 0.5, 1, \tan^{-1} b_i\right]$.

***Proposition 2*:** System (A_2) is a complement of System (10).

In System (1) if b_i follows a general form of distributions $F(b_i) = B_i e^{-A_i h(b_i)}$ then with the similar argument of Case 2, we shall obtain the following System (A_2)

$$
\begin{aligned}
&Max\, R_s(X) = \prod_{j=1}^{n}\left[1-(1-r_j)^{x_j}\right] \\
&subject\ to \\
&\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \le 1-p \\
&x_j \ge 1, \quad j = 1,2,\ldots n.
\end{aligned}
\tag{A_2}
$$

After inserting the particular value of the parameters $[A_i, B_i, C_i, h(b_i)]$ in the above deterministic constraints, we get different deterministic constraints for the various distributions listed below: Inverse Weibull distribution $\left[\theta_i, 1, b_i^{a_i}\right]$, Power Function distribution $\left[-a_i, \lambda_i^{a_i}, \ln(b_i)\right]$, Logistic distribution $\left[1,1,\ln\left(1+e^{-b_i}\right)\right]$, Burr Type II distribution $\left[\theta_i, 1, \ln\left(1+e^{-b_i}\right)\right]$, Burr Type III distribution $\left[k_i, 1, \ln\left(1+b_i^{-\lambda_i}\right)\right]$, Burr Type IV distribution $\left[k_i, 1, \ln\left\{1+\left\{b_i^{-1}(\lambda_i - b_i)\right\}^{1/\lambda_i}\right\}\right]$, Burr Type V distribution $\left[k_i, 1, \ln\left(1+\lambda_i e^{-\tan b_i}\right)\right]$, Burr Type VI distribution $\left[k_i, 1, \ln\left(1+\lambda_i e^{-k_i \sinh b_i}\right)\right]$, Burr Type VII distribution $\left[-k_i, 1, \ln(0.5+0.5\tanh b_i)\right]$, Burr Type VIII distribution $\left[-k_i, 1, \ln\left(2\pi^{-1}\tan^{-1} e^{b_i}\right)\right]$, Burr Type X distribution $\left[-k_i, 1, \ln\left(1-e^{-b_i^2}\right)\right]$, Burr Type XI distribution $\left[-k_i, 1, \ln\left\{b_i - (2\pi)^{-1}\sin 2\pi b_i\right\}\right]$, Gumble distribution $\left[1,1,e^{-b_i}\right]$, and Extreme Value II distribution $\left[\theta_i^{a_i}, 1, b_i^{-a_i}\right]$.